# Econ 802
# Final Exam Answers

Greg Dow December 2017

1. (a) Convexity of the input requirement set is equivalent to quasi-concavity of the production function. This is useful in cost minimization problems because it implies that whenever the first order conditions are satisfied, we have a solution. Thus there is no need to check second order conditions. Similarly, convexity of the upper contour set is equivalent to quasi-concavity of the utility function. This is useful when maximizing utility subject to a budget constraint because it implies that whenever the first order conditions are satisfied we have a solution. Again, there is no need to check second order conditions.

(b) When firms are price-takers, increasing returns to scale (globally) typically implies that the profit maximization problem has no solution. For example if there is some input vector at which the firm has positive profit, the firm can always get a larger profit by increasing its output. This simultaneously increases profit per unit (due to decreasing average cost) and increases the number of units sold. Increasing returns is not interesting in consumer theory because any positive increasing transformation of a utility function gives a new utility function that represents the same preference ordering. Such transformations can change the utility function from increasing returns to decreasing returns or vice versa, without having any effect on behavior.

(c) The Slutsky equation arises from the assumption that a consumer maximizes utility subject to a budget constraint. In order to get a similar equation for the firm, we would need to assume that the firm maximizes output subject to a cost or expenditure constraint. Economists do not generally believe this is a good description of firm behavior, and prefer models based on profit maximization. But note that there is a parallel between cost min for a firm and expenditure minimization for a household or consumer - these are essentially the same mathematical problem.

2. (a) The overall problem $\max p[ax_1^{1/2} + bx_2] - w_1 x_1 - w_2 x_2$ can be broken into two sub-problems:

$$\max pax_1^{1/2} - w_1 x_1 \quad \text{and} \quad \max (pb - w_2)x_2$$

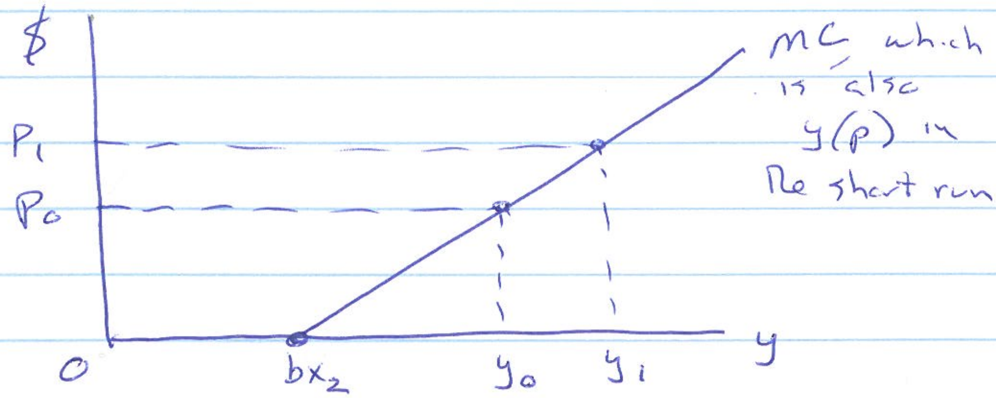
The derivative in the first case is

$\frac{pa}{2} x_1^{-1/2} - w_1$. This $\to +\infty$ as $x_1 \to 0$, so there is never a boundary solution with $x_1 = 0$.

Also, the second derivative is strictly negative everywhere, so the sufficient SOC will always hold. There is a unique $x_1 > 0$ satisfying the FOC: $x_1 = \left(\frac{pa}{2w_1}\right)^2$.

For the second sub-problem, there is a unique solution $x_2 = 0$ when $pb - w_2 < 0$, and any $x_2 \geq 0$ is a solution when $pb - w_2 = 0$. There is no solution when $pb - w_2 > 0$.
Summary: The overall problem has a solution iff $pb \leq w_2$. The solution is unique iff $pb < w_2$. It never involves $x_1 = 0$, always involves $x_2 = 0$ when $pb < w_2$, and could involve $x_2 = 0$ when $pb = w_2$.

(b) We have $y = a x_1^{1/2} + b x_2 \Rightarrow x_1 = \left(\frac{y - b x_2}{a}\right)^2$
Note that the firm always gets an output of at least $b x_2$, so only the case $y \geq b x_2$ is interesting here. The solution for $x_1$ gives $c(w, y, x_2) = w_1 \left(\frac{y - b x_2}{a}\right)^2 + w_2 x_2$.
The FOC for profit max is
$p = MC = \frac{2 w_1}{a^2}(y - b x_2)$. Note: MC rising $\Rightarrow$ SOC holds. It is not hard to show that this exceeds $AVC = \frac{w_1}{y}\left(\frac{y - b x_2}{a}\right)^2$ for all $y$, so we can ignore the shutdown issue. $P = MC$ gives the short run supply function $y_{SR}(p) = \frac{p a^2}{2 w_1} + b x_2$.
On a graph:




Thus the SR supply curve has a positive horizontal intercept and the firm produces $b x_2 > 0$ even at $P = 0$. For higher prices the supply curve is linear.

2 (c) We need to solve $\min w_1 x_1 + w_2 x_2$ subject to $y = a x_1^{1/2} + b x_2$. The constraint gives $x_2 = \frac{y - a x_1^{1/2}}{b}$. It is easier to substitute this into the objective function than to set up a Lagrangean (due to quasi-linearity!) So we choose $x_1$ to solve $\min w_1 x_1 + w_2 \left[\frac{y - a x_1^{1/2}}{b}\right]$. The derivative with respect to $x_1$ is

$$w_1 - \frac{w_2 a}{b}\left(\frac{1}{2}\right) x_1^{-1/2}$$

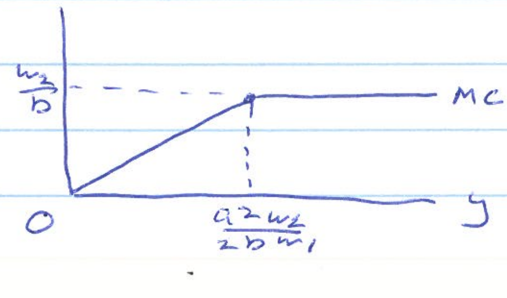
If there is an interior solution, we set this equal to zero to get $x_1 = \left(\frac{a w_2}{2 b w_1}\right)^2$. This gives $x_2 = \frac{y}{b} - \frac{a}{b} x_1^{1/2}$

$\Rightarrow x_2 = \frac{y}{b} - \frac{a^2 w_2}{2 b^2 w_1}$. But we have to be careful not to violate the non-negativity constraint $x_2 \geq 0$. This is not a problem as long as $y \geq \frac{a^2 w_2}{2 b w_1}$. But if $y < \frac{a^2 w_2}{2 b w_1}$ the firm produces all of its output using only $x_1$, with $x_2 = 0$ (in this case the derivative of the objective with respect to $x_1$ is negative - if the constraint $x_2 = 0$ did not bind, the firm would increase $x_1$ further to reduce cost more). So the LR cost function is

$$c(w, y) = \frac{w_1 y^2}{a^2} \quad \text{if} \quad y \leq \frac{a^2 w_2}{2 b w_1} \quad \text{and}$$

$$c(w, y) = w_1 \left(\frac{a w_2}{2 b w_1}\right)^2 + w_2 \left[\frac{y}{b} - \frac{a^2 w_2}{2 b^2 w_1}\right] \quad \text{if } y \geq \frac{a^2 w_2}{2 b w_1}$$

On a graph:
initially $MC = \frac{2 w_1 y}{a^2}$
Then $MC = \frac{w_2}{b}$




(This is also the supply curve)

Mathematically, the supply function is

$$y_{LR}(p) = \frac{pa^2}{2w_1} \quad \text{for } p \le \frac{w_2}{b}$$

$$y_{LR}(p) = \text{any } y \ge \frac{a^2 w_2}{2bw_1} \quad \text{for } p = \frac{w_2}{b}$$

$$y_{LR}(p) \text{ undefined} \quad \text{for } p > \frac{w_2}{b}.$$

Note: it can be shown that $MC > AC$ for all $y > 0$, so the firm always gets positive profit when $p = MC$.

3. (a) Set up the Lagrangean

$$L = \frac{1}{n} \sum_i \ln x_i - \lambda \left[ \sum p_i x_i - m \right]$$

FOC: $\frac{1}{n x_i} = \lambda p_i$ for all $i$

$$\Rightarrow \frac{1}{n} = \lambda p_i x_i \Rightarrow 1 = \lambda \sum p_i x_i = \lambda m$$

$$\Rightarrow \lambda = \frac{1}{m}$$

$$\Rightarrow x_i = \frac{1}{n \lambda p_i} \quad \text{so} \quad \boxed{x_i(p, m) = \frac{m}{n p_i} \text{ for all } i.}$$

(b) To get $v(p,m)$, substitute Marshallian demands into the direct utility function $\Rightarrow$

$$v(p,m) = \frac{1}{n} \sum_i \ln\left(\frac{m}{n p_i}\right) = \frac{1}{n} \sum_i \left[ \ln\left(\frac{m}{n}\right) - \ln p_i \right]$$

$$\Rightarrow v(p,m) = \ln(m) - \ln(n) - \frac{1}{n} \sum_i \ln p_i.$$

To get $e(p,u)$, we need to invert this. Write $u = v(p,m)$ and $m = e(p,u)$, and solve for $m$:

$$u = \ln(m) - \ln(n) - \frac{1}{n} \sum_i \ln p_i$$

$$\Rightarrow \ln(m) = u + \ln(n) + \frac{1}{n} \sum_i \ln p_i$$

$$\Rightarrow m = e(p,u) = e^u n (p_1 p_2 \cdots p_n)^{1/n}.$$

3 (c) Suppose $x = \left(\frac{m}{np_1}, \frac{m}{np_2}, \dots \frac{m}{np_n}\right)$. Clearly $x$ is optimal for the prices $p$ and income $m$ (see part (a)), so $x \in IEP$. It is also clear that $tx$ will be demanded at income $tm$ for the same price vector $p$, so $tx \in IEP$.

The question is why this happens. The utility function $u(x)$ is a log transformation of the Cobb-Douglas utility function $g(x) = (x_1 x_2 \dots x_n)^{1/n}$. Because utility is ordinal and the log transformation is increasing, $g(x)$ represents the same preferences as $u(x)$. Also, $g(x)$ is linearly homogeneous (and thus $u(x)$ is homothetic). A property of LH functions is that their derivatives are homogeneous of degree zero. Now suppose $x$ satisfies all of the FOC and thus is optimal (recall that the FOC are sufficient here). This implies

$$\frac{\partial g(x)}{\partial x_i} = \lambda p_i \text{ for all } i \text{ and } px = m.$$

By homogeneity it is also true that

$$\frac{\partial g(tx)}{\partial x_i} = \lambda p_i \text{ for all } i \text{ and } p(tx) = tm.$$

So $tx$ satisfies all FOC at income $tm$ and must be optimal for $(p, tm)$.

4 (a) We need to solve max $\ln x_{A1} + \ln x_{A2}$ subject to

$$x_{A1} + x_{B1} = 1, \quad x_{A2} + x_{B2} = 1, \text{ and } k_1 x_{B1} + k_2 x_{B2} = u_B^0.$$

Use the physical constraints to write $k_1(1-x_{A1}) + k_2(1-x_{A2}) = u_B^0$.
Write the Lagrangean as

$$L = \ln x_{A1} + \ln x_{A2} - \lambda\left[u_B^0 - k_1(1-x_{A1}) - k_2(1-x_{A2})\right]$$

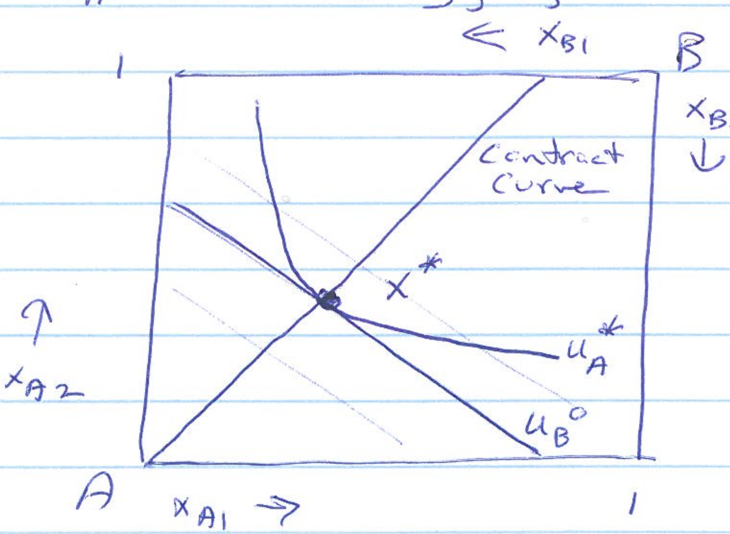
$$\text{FOC: } \left.\begin{aligned} \frac{1}{x_{A1}} &= \lambda k_1 \\ \frac{1}{x_{A2}} &= \lambda k_2 \end{aligned}\right\} \Rightarrow \frac{x_{A2}}{x_{A1}} = \frac{k_1}{k_2}$$

$$\Rightarrow u_B^0 = k_1(1-x_{A1}) + k_2\left[1 - \frac{k_1 x_{A1}}{k_2}\right] = k_1 + k_2 - 2k_1 x_{A1}$$

$$\Rightarrow x_{A1} = \frac{k_1 + k_2 - u_B^0}{2k_1} \qquad x_{A2} = \frac{k_1 + k_2 - u_B^0}{2k_2}$$

<u>Note</u>: due to $0 \le x_{A1} \le 1$ and $0 \le x_{A2} \le 1$, this only makes sense when $k_2 - k_1 \le u_B^0$, $k_1 - k_2 \le u_B^0$, and $u_B^0 \le k_1 + k_2$.

<u>Also note</u>: when $u_B^0$ is larger, $x_{A1}$ and $x_{A2}$ are smaller. The consumption bundle $x_B$ is easily obtained by subtracting $x_A$ from the aggregate endowments.




B's indiff curves are linear with slope $-\frac{k_1}{k_2}$. A's indiff curves are strictly convex. The solution is a point like $x^*$ where $MRS_A = MRS_B$. The contract curve is the ray with $\frac{x_{A2}}{x_{A1}} = \frac{k_1}{k_2}$ (the set of Pareto efficient allocations)

4(b) $\max\ a[\ln x_{A1} + \ln x_{A2}] + b[k_1 x_{B1} + k_2 x_{B2}]$

Again use $x_{B1} = 1 - x_{A1}$ and $x_{B2} = 1 - X_{A2}$ so we solve

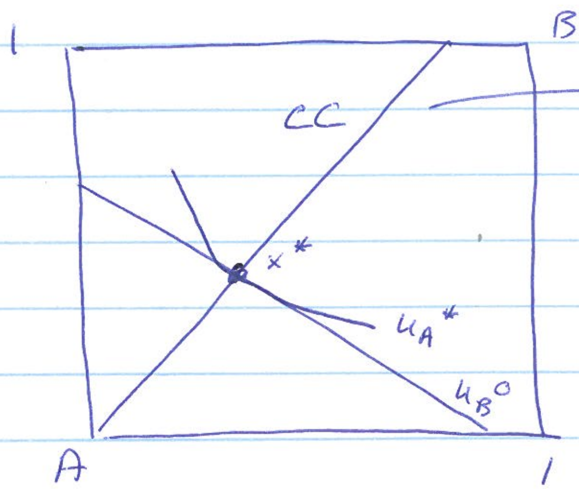
$$\max\ a[\ln x_{A1} + \ln x_{A2}] + b[k_1(1-x_{A1}) + k_2(1-x_{A2})]$$

FOC: $\frac{a}{x_{A1}} = bk_1$ and $\frac{a}{x_{A2}} = bk_2 \Big\} \Rightarrow \frac{x_{A2}}{x_{A1}} = \frac{k_1}{k_2}$ (as before)

To get the same allocation as in part (a), we need

$$x_{A1} = \frac{a}{bk_1} = \frac{k_1 + k_2 - u_B^0}{2k_1} \Rightarrow \frac{a}{b} = \frac{k_1 + k_2 - u_B^0}{2}$$

Note that for larger $u_B^0$, we need a smaller ratio $a/b$, ie the planner puts less weight on $u_A$. The absolute levels of a and b are unimportant, only the ratio matters.




The planner's solution for any $a/b$ must be Pareto efficient so it must be on the contract curve with $\frac{x_{A2}}{x_{A1}} = \frac{k_1}{k_2}$. The ratio $a/b$ determines which point on CC is chosen. Higher values of $a/b$ put A on higher indiff curves, and B on lower indiff curves.

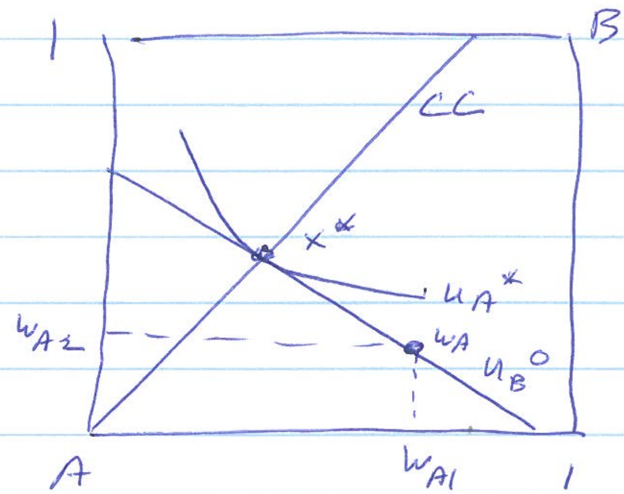
(c) The prices must satisfy $\frac{p_1}{p_2} = \frac{k_1}{k_2}$ so that A's budget line has the same slope as B's indifference curves, because WE allocations must be on the contract curve due to the First Welfare Theorem.

In WE, A's bundle solves max $\ln x_{A1} + \ln x_{A2}$ subject to $p_1 x_{A1} + p_2 x_{A2} = p_1 w_{A1} + p_2 w_{A2}$ $(= m_A)$. Thus A's endowment point and optimal bundle must be on the same budget line. Using the same values of $x_{A1}$ and $x_{A2}$ as in parts (a) and (b), and the fact that $\frac{p_1}{p_2} = \frac{k_1}{k_2}$, we obtain

$$\frac{k_1}{k_2}\left[\frac{k_1 + k_2 - u_B^0}{2k_1}\right] + \left[\frac{k_1 + k_2 - u_B^0}{2k_2}\right] = \frac{k_1}{k_2} w_{A1} + w_{A2}$$

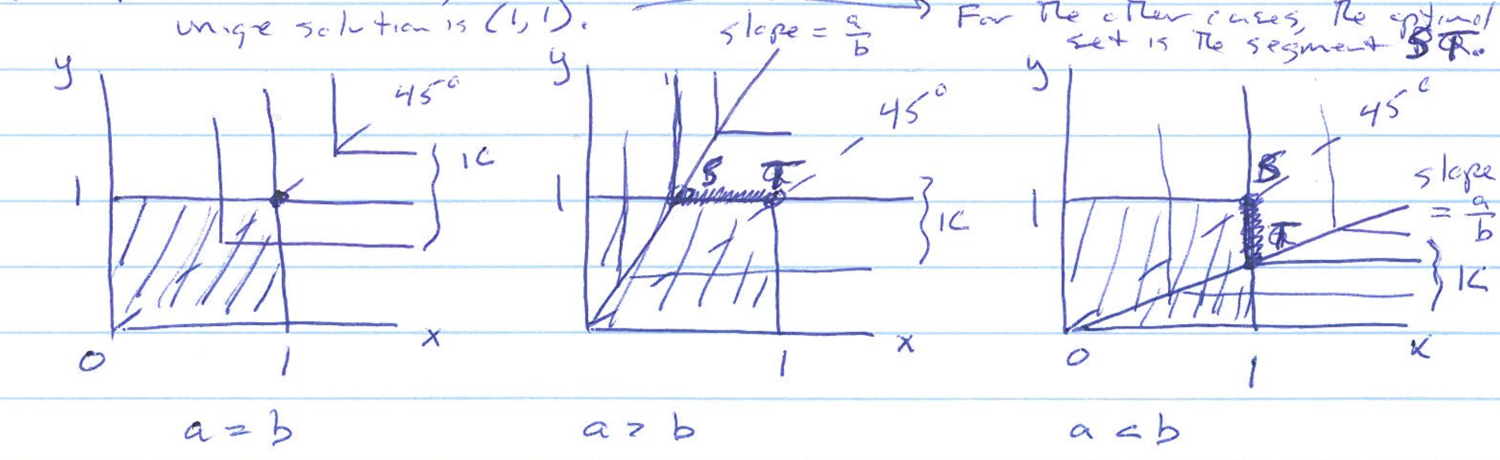
$$\Rightarrow \quad k_1 + k_2 - u_B^0 = k_1 w_{A1} + k_2 w_{A2}. \qquad \underline{\text{or}}: u_B^0 = k_1 w_{B1} + k_2 w_{B2}.$$

Any individual endowment point $w_A = (w_{A1}, w_{A2})$ that satisfies this condition will work.



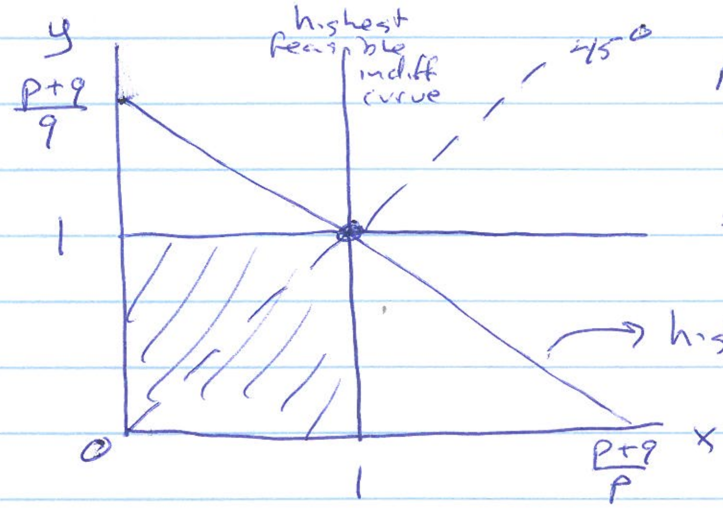

One example of such a point is $w_A$ in the graph. Note that B is indifferent between the allocations at $w_A$ and $x^*$. However, A prefers to be at $x^*$ and B is willing to make the necessary trades.

5 (a) In each case, the shaded area is the feasible set. For $a=b$ the unique solution is (1, 1). For the other cases, the optimal set is the segment $\overline{ST}$.



5 (b) Assume $a = b$, let $p > 0$, $q > 0$.

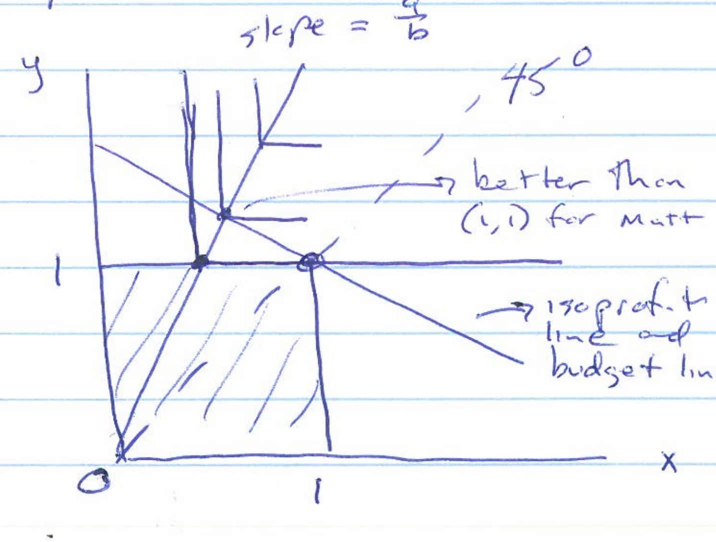
The firm has profit $px + qy$, which is uniquely maximized at $x = y = 1$. The resulting profit is $p + q$. Therefore Matt's income is also $p + q$. His budget constraint is $px + qy \leq p + q$. Clearly he can afford the bundle $x = y = 1$. The budget line has the slope $-\frac{p}{q}$ which is the same as the slope of the firm's isoprofit lines. Any consumption bundle $(x, y) \neq (1, 1)$ along the budget line is on a lower indifference curve. Thus Matt's utility is uniquely maximized at $x = y = 1$. Both markets clear because the firm produces one unit of each good and Matt demands one unit of each good. There is no excess supply (This is the only WE allocation.)




Note: isoprofit lines have $\pi = px + qy \Rightarrow y = \frac{\pi - px}{q}$
$\Rightarrow$ slope $= -\frac{p}{q}$.

(c) Assume $a > b$.

If $p > 0$, $q > 0$ then firm goes to $(1,1)$ as before. But now Matt can get to a higher indiff curve by moving to the ray with slope $\frac{a}{b}$
$\Rightarrow$ excess demand for y
$\Rightarrow$ not WE.

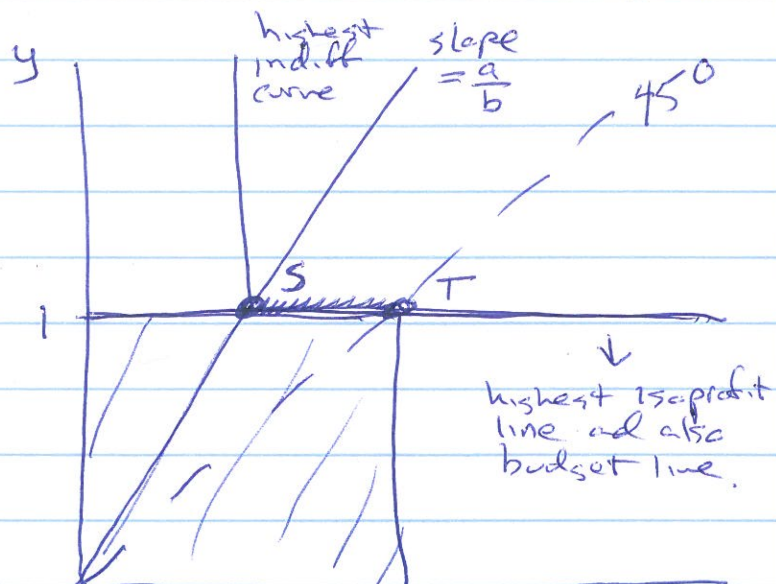


5 (c) (continued)

if $p = q = 0$, the firm is indifferent among all $(x, y)$ with $0 \le x \le 1$ and $0 \le y \le 1$. Its profit must be zero. Matt gets no income from the firm, but on the other hand, goods have zero prices, so he has no budget constraint. In this case his utility max problem has no solution ⇒ wants an infinite amount of both goods ⇒ no WE.

When $a > b$ there is a WE with $p = 0$ and $q > 0$. Profit for the firm is $qy$ which is maxed at $y = 1$. The firm is indifferent to all $0 \le x \le 1$. Its profit $= q$. Note that the slope of its isoprofit line is $-\frac{p}{q} = 0$, so isoprofit lines are horizontal.

Matt's income is $q$ and his budget line is also horizontal due to $p = 0$. He can afford to buy $y = 1$ and any $x \ge 0$. So his budget line is the entire horizontal line with $y = 1$ and $x \ge 0$.




Any point on the line segment ST both maxes profit for the firm and maxes utility for Matt. At any such point, both markets clear and we have WE. We can also have WE with excess supply of x if Matt chooses a point on ST that is to the left of the point chosen by the firm. → (Matt cannot choose a point to the right: this ⇒ excess demand)

↳ This is OK because x has a zero price. Note that the market for y always clears exactly.